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EXAMPLE OF STOCHASTIC BIFURCATION IN THE THEORY OF FLEXURE OF IMPERFECT PLATES*

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Flexural stability of thin, rectangular (in the plane) hinged plates is investigated for the case when the initial deflections of their middle surfaces from their perfect shape, form an ensemble with quasigaussian probability measure, and the total deflection is described by the Kármán-type equations /1/. The external normal forces and displacements of the point on the contour of the middle surface are assumed given. A functional method (**) is used, based on the following: a probability functional which is a generalization of the probability density, induced by the measure of probability of the initial flexures and by the operator of the problem, is constructed on the set of possible forms of flexure. The instability is related to the branching of the modes of this functional. Additional assumptions of smallness of the dispersion and correlation scales of the initial deflections are also made. The Galerkin method is used to determine small solutions of the equation at the extremal of the probability functional. Simple relations are derived, which can be used to find the values of the loads corresponding to the instances at which these solutions branch off from the trivial solution. It is shown that up to the instant of first branching the trivial extremal is the only mode of the probability functional. Beginning from that instant, the probability functional attains its maxima at other extremals which have branched off the trivial extremal, and these extremals are regarded as the fundamental forms of loss of stability.

1. We consider a thin elastic plate with random initial deflections of its middle surface from the perfect form. We assume that at any instant of the loading the hinged edges of the plate lie in the plane of the supporting contour. Let the plate occupy (in the plane) a rectangular area D in R^2, with sides of length a and b. We direct the coordinate axes Ox and Oy along these sides, and apply to the plate a variant of the nonlinear theory of shells based on the Kirchhoff–Love hypothesis /1/. The boundary conditions corresponding to this variant include the conditions for the external forces and displacements of the points on the contour of the middle surface. We assume that the plate is compressed by an external force $p > 0$ directed along the Ox-axis. We specify the displacements of the points on the middle surface contour along the straight lines $x = 0, a$, in the form of a linear function of the coordinate y with angular coefficient $\mu (p/E)$, and along the straight lines $y = 0, b$ as a linear function of the coordinate x with angular coefficient p/E (μ is the Poisson's ratio and E is the modulus of elasticity).

We write the stress function at the middle surface in the form of a difference

$$\Phi_0 (x, y) = \Phi_1 (x, y) - px^2/2 \tag{1.1}$$

According to the boundary conditions adopted for the forces and displacements, the function Φ_1 and its second order derivatives with respect to the inward normal ρ to ∂D, all vanish on ∂D (∂D is the boundary of the region D). Let us eliminate the stress function (1.1) from the Kármán-type equations for the plate with initial flexure $\omega_0 (r) = \omega_0 (x, y)$ /1/. For the function of the total flexure $\omega (r) = \omega (x, y)$ we obtain

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**) Volkov S.I. Branching of the extremals of the probability functional and stability of a nonlinear elastic rod with random form distortions. Novosibirsk, Dep. v VINITI, 23.1.1979.

$$N[r;\omega,\omega_0]\,\omega(r) = \Delta^2\omega_0(r) \tag{1.2}$$

$$N[r;\omega,\omega_0]\,\varphi(r) = \left(\Delta^2 + \frac{hp}{E}\frac{\partial^2}{\partial x^2}\right)\varphi(r) + \frac{hE}{d}\{[\varphi(r), \Phi[r;\omega] - \Phi[r;\omega_0]]\}$$

$$\Phi[r;u] = \int_D dr_1 g(r,r_1)[u(r_1), u(r_1)]$$

$$[u(r), \varphi(r)] = u_{xx}\varphi_{yy} + u_{yy}\varphi_{xx} - 2u_{xy}\varphi_{xy}$$

Here $N[r;\omega,\omega_0]$ is a nonlinear operator, Δ^2 is a biharmonic operator, $g(r,r_1)$ is the kernel of the Green's operator inverse to Δ^2 in D under the boundary conditions analogous to the boundary conditions for Φ_1, $dr = dxdy$ is an area element on D, h is the plane thickness, d is cylindrical rigidity and the expression $F[r;u]$ means that F is a function of the coordinates x, y and a functional of the field $u(r) = u(x,y)$.

Let the probability measure of the initial flexures ω_0 resemble the Gaussian measure in the sense of representing the higher correlation moments in terms of the lower moments. We assume that the random field $\omega_0(r)$ has zero average value, low dispersion and small correlation scales correlated with each other in such a manner that the requirement of the root mean square shallowness of the initial middle surface is fulfilled. Then from the Chebyshev inequality we have, probabilistically,

$$\Phi[r;\omega_0] \simeq \langle\Phi[r;\omega_0]\rangle \tag{1.3}$$

Here the square brackets denote averaging over the ensemble of realizations of the initial deflections ω_0.

Let us replace, in the second formula of (1.2), the function $\Phi[r;\omega_0]$ by its approximate value according to (1.3). This yields an expression defining a quasiergodic approximation of the operator $N[r;\omega,\omega_0]$. We denote this approximation by $N[r,\omega]$. For the total flexures ω the first equation of (1.2) assumes, after replacing $N[r;\omega,\omega_0]$ by $N[r;\omega]$, the form of an equation with a stochastic source in the right-hand side

$$N[r;\omega]\,\omega(r) = \Delta^2\omega_0(r) \tag{1.4}$$

We supplement this equation with the boundary conditions on the contour ∂D, following from the assumption that the plate is hinged

$$\omega = \partial^2\omega/\partial\rho^2 = 0 \tag{1.5}$$

and investigate the stochastic stability of the imperfect plates with help of the nonlinear equations (1.4) and boundary conditions (1.5).

2. Let $\omega[r;\omega_0]$ be a solution of (1.4),(1.5), functionally dependent on the concrete realization of the initial flexure $\omega_0(r)$. We introduce the characteristic functional

$$\Psi[\theta] = \langle\exp(i\int_D dr\omega[r;\omega_0]\,\theta(r))\rangle \tag{2.1}$$

and the linear response functional

$$G[r,r';\theta] = \left\langle\left(\frac{\delta}{\delta\chi(r')}\omega[r;\omega_0+\chi]\right)\Big|_{\chi=0}\exp\left(i\int_D dr\omega[r;\omega_0]\,\theta(r)\right)\right\rangle \tag{2.2}$$

Here $\chi(r)$ is the deterministic additional to $\omega_0(r)$ in the right-hand side of (1.4). The similarity between the probabilistic measure of the deflections ω_0 and the Gaussian measure makes it possible to obtain from (1.4), using the known methods /2,3/, a closed system of equations in variational derivatives connecting the functionals (2.1) and (2.2). Let Ω denote the set of possible forms of flexure with given boundary conditions (1.5). We shall represent an element of this set by a Fourier series in eigenfunctions $f_{kj}(r)$ of the linearized (at $\omega_0 = 0$) boundary value problem (1.2), (1.5)

$$\omega(r) = \sum_k\sum_j \omega_{kj} f_{kj}(r) \tag{2.3}$$

The characteristic function $\Psi^{(n\times m)}$ of the coefficients ω_{kj} $(k = 1,\ldots,n;\ j = 1,\ldots,m)$ has, by definition, the form of a Fourier integral of the compatible $n\times m$-dimensional probability density $P^{(n\times m)}$ of the quantitites ω_{kj}. Here the role of integration measure is played by the "volume" element $dV^{(n\times m)}$ of the possible values of the coefficients ω_{kj} in the Euclidean space $R^{n\times m}$.

We assume that there exists, if only formally, a limit of the sequence of such integrals when $n\to\infty$ and $m\to\infty$. Let us change the integration measures at the instant of the passage to the limit. Using the linear transformation (2.3), we pass from the integration over the measure in the space of possible values of the coefficients ω_{kj}, to the integration over another measure, namely an analog of "volume" in the space of functions $\omega(r)$ of Ω defined on D. For the change of the measures (variables) of integration at the instant when the quantity $\Psi^{(n\times m)}$ attains its formal limit as $n\to\infty, m\to\infty$, we have the corresponding passage from the finite-dimensional probability density $P^{(n\times m)}$ of the quantities ω_{kj}, to its infinite-dimensional generalization represented by the probability functional $P[\omega]$ defined on the set Ω. Let us denote the above measure in the space of functions $\omega(r)$ in Ω by $\mu[\omega(r)\in\Omega]$. Then we have the corresponding symbolic equation

$$dV^{(\infty\times\infty)}P^{(\infty\times\infty)} = d\mu[\omega(r)\in\Omega]\,P[\omega]$$

Using this equation we can write the formal limit of the sequence of the quantities $\Psi^{(n\times m)}$ in the form of the following continuous integral

$$\Psi[\theta]=\int d\mu[\omega(r)\in\Omega]\left\{P[\omega]\times\exp\left(i\int_D dr\,\omega(r)\,\theta(r)\right)\right\}\quad (\Psi[\theta]=\Psi^{(\infty\times\infty)}) \tag{2.4}$$

The integral (2.4) represents the infinitely-dimensional Fourier transform of the generalized probability density $P[\omega]$ of the random field $\omega(r)\in\Omega$ (of the generalized random process according to he terminology of /4/) and is used, together with the expression (2.1), as the means of determining $P[\omega]$. Below we show that the convergence of the integral (2.4) depends on the stability of the initial physical system, just as the continuous integrals of the Bose quasiprobabilistic theories /5/.

Applying the transformation (2.4) to the equations for the functionals $\Psi[\theta]$ and $G(r, r'; \theta)$, we obtain the following closed system of equations for determining the probability functional $P[\omega]$:

$$N[r;\omega]\,\omega(r)\,P[\omega]=-\int_D dr_1\int_D dr_2\left\{F(r,r_1)\times\frac{\delta}{\delta\omega(r_2)}(\Gamma[r_2,r_1;\omega]\,P[\omega])\right\} \tag{2.5}$$

$$N[r;\omega]\,\Gamma[r,r';\omega]+\int_D dr_1\left\{\frac{\delta N[r;\omega]}{\delta\omega(r_1)}\Gamma[r_1,r';\omega]\right\}=\delta(r,r')$$

$$F(r,r')=\int_D dr_1\int_D dr_2\,\{g^{-1}(r,r_1)\,K(r,r_2)\,g^{-1}(r_2,r')\}$$

$$K(r,r')=\langle\omega_0(r)\,\omega_0(r')\rangle$$

Here $\Gamma[r,r';\omega]$ is the response function to an infinitesimal load added to the right-hand side of (1.3), and $\delta(r,r')$ is the δ-function defined in D and connected with the boundary conditions (1.5). The expressions (2.5) should be supplemented by the normalizing and non-negativity conditions

$$\int d\mu[\omega(r)\in\Omega]\,P[\omega]=1,\qquad P[\omega]\geqslant 0 \tag{2.6}$$

Let the integral operation with the kernel $F(r,r')$ defined on the set $W\supset\Omega$ of functions given in D and satisfying the conditions (1.5), be nondegenerate. Then the equations (2.5) have the following functional on Ω, whose solution satisfies the conditions (2.6):

$$P[\omega]=C\,|\operatorname{Det}[\Gamma^{-1}(\omega)]|\times\exp\left\{-\frac{1}{2}\int_D dr\int_D dr_1\,[(N[r;\omega]\,\omega(r))\,F^{-1}(r,r_1)\,(N[r_1;\omega]\,\omega(r_1))]\right\} \tag{2.7}$$

Here $\operatorname{Det}[\Gamma^{-1}(\omega)]$ is the Fredholm determinant of the integral operation with kernel $\Gamma^{-1}[r,r';\omega]$ defined on W. The constant C can be found from the first condition of (2.6). The condition of local extremum demands that the variational derivative of the functional $P[\omega]$ vanishes on the extremal $\omega=\omega_*$. Equating such a derivative in the first expression of (2.5) to zero, we obtain the equation for the functional (2.7) on the extremal $\omega_*(r)$

$$N[r;\omega_*]\,\omega_*(r)=\int_D dr_1\ldots\int_D dr_4\left\{F(r,r_1)\times\Gamma[r_2,r_3;\omega_*]\left(\frac{\delta}{\delta\omega(r_2)}\Gamma^{-1}[r_3,r_4;\omega]\right)\Big|_{\omega=\omega_*}\Gamma[r_4,r_1,\omega_*]\right. \tag{2.8}$$

The right-hand side of this equation contains a real nonanalytic operation with kernal $\Gamma[r, r'; \omega_*]$. However, from (2.7) it follows that the matrix $\Gamma^{-1}[r, r'; \omega_*]$ degenerates only in those realizations of its functional argument ω_* of Ω, which have zero probability. The operators, the kernals of which form the matrices $\Gamma[r, r'; \omega_*]$, are bounded in probability; and so are the derivatives of the right-hand side of (2.8) with respect to the functional argument $\omega_*(r)$ or some other parameter of the problem.

3. Let us write $p = p_0 + \gamma$. Let

$$\mathrm{Det}\,[\Gamma^{-1}(\omega)]\,|_{\omega=0} \neq 0$$

in the small neighborhood of p_0. Then the terms of the equation (2.8) can be expanded in powers of γ and $\omega_*(r)$ with nonzero probability. Taking into account the principal terms of this expansion only, we obtain (using the notation $u(r) = \omega_*(r)$):

$$B(r, p_0)\,u(r) = H[r; u] + \ldots \tag{3.1}$$

$$H[r; u] = -\frac{\gamma h}{d}u_{xx}(r) - \frac{hE}{d}[u(r), \Phi[r; u]]$$

$$B(r, p_0)\,u(r) = \left(\Delta^2 + \frac{p_0 h}{d}\frac{\partial^2}{\partial x^2}\right)u(r) - \frac{hE}{d}[u(r), \langle\Phi[r; \omega_0]\rangle] - \int\limits_D dr_1 \ldots \int\limits_D dr_5 F(r, r_1)\{\Gamma_0(r_4, r_1, p_0) \times$$

$$\Gamma_1(r_3, r_4, r_2, r_5)\,u(r_5)\,\Gamma_0(r_4, r_1, p_0)\}$$

Here $B(r, p_0)$ is a linear operator, and Γ_1 is a vertical kernel determined from the expression

$$\frac{\delta}{\delta\omega(r_2)}\Gamma^{-1}[r, r_1; \omega] = \int\limits_D dr_3 \Gamma_1(r, r_1, r_2, r_3)\,\omega(r_3)$$

The function $\Gamma_0(r, r', p_0)$ is equal to $\Gamma[r, r'; \omega = 0]$ for $p = p_0$ and is, in general, bounded probabilistically. Let us denote by $\varphi_m(r)$ the zeros of the operator $B(r, p_0)$. Under the conditions (1.5) the zeros will be represented by the nontrivial solutions of the linear equation

$$B(r, p_0)\,u(r) = 0 \tag{3.2}$$

We denote by p_{0m} the values of the parameter p_0 corresponding to the instances of noninvertibility of the operator $B(r, p_0)$. When $K(r, r')$ are sufficiently small, the problem (3.2), (1.5) admits, within the framwork of the Galerkin method, the approximate solutions $\varphi_m(r) \approx f_{kj}(r)$ where

$$f_{kj}(r) = 2(ab)^{-1/2}\sin\left(\frac{\pi k x}{a}\right)\sin\left(\frac{\pi j y}{b}\right)$$

The instances of noninvertibility corresponding to these solutions occur when $p_0 = \Lambda_{kj}$ where

$$\Lambda_{kj} = \frac{d}{h}\left(\frac{a}{\pi k}\right)^2(g_{k,j}^{-1} - \alpha_1\Omega_{k,j}^{(1)} - \alpha_2\Omega_{k,j}^{(2)}) \tag{3.3}$$

$$g_{x,y} = \left[\left(\frac{\pi x}{a}\right)^2 + \left(\frac{\pi y}{b}\right)^2\right]^{-2}$$

$$\alpha_1 = \frac{16\pi^2 hE}{d\sqrt{(ab)^5}}, \quad \alpha_2 = \left[\frac{3\pi^4 hE}{4d(ab)^5}\right]^{1/2}$$

$$\Omega_{k,j}^{(1)} = (kj)^2\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\{g_{2s+1,2t+1}M_{2s+1,2t+1}T_{2s+1,2t+1,k,j}\}$$

$$\Omega_{k,j}^{(2)} = \left\{(kj)^4 K_{kj,kj}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}g_{2s+1,2t+1}T^2_{2s+1,2t+1,k,j}\right\}^{1/2}$$

$$T_{x,y,v,t} = \frac{y}{x(4t^2 - y^2)} + \frac{x}{y(4v^2 - t^2)}$$

$$M_{k,j} = \int\limits_D dr f_{kj}(r)\langle[\omega_0(r), \omega_0(r)]\rangle$$

The quantities $K_{kj,kj}$ represents the coefficients of expansion of the correlation function $K(r, r')$ of the initial deflections, diagonal with respect to the pair (k, j) of indices, over the system $\{f_{kj}(r)\}$.

Let the quantity $p_0 = p_{0m} \approx \Lambda_{kj}$ have only a single corresponding zero of the operator $B(r, p_0)$. Then in analogy with the nonlinear equations with operators in Banach spaces /6/, the first equation of (3.1) admits the solution

$$u(r) \approx \eta f_{kj}(r) + R_{kj}^{-1} \int_D dr_1 g(r, r_1) H[r_1; u] \qquad (3.4)$$
$$\eta = \int_D dr u(r) f_{kj}(r)$$

where R_{kj}^{-1} is a Green's operator which can be transformed, under the conditions (1.5), to its inverse R_{kj}. The operator R_{kj} is given by the expression

$$R_{kj} u(r) = \int_D dr_1 [(f_{kj}(r) f_{kj}(r_1) + g(r, r_1) B(r_1, p_0 = \Lambda_{kj})) u(r_1)]$$

Following /6/, we construct for the parameter η, so called branching equations

$$L_{kj}^{(1)}(p - \Lambda_{kj}) + L_{kj}^{(2)} \eta^2 \approx 0 \qquad (3.5)$$
$$L_{kj}^{(1)} = -\frac{h}{d} \int_D dr f_{kj}(r) \frac{\partial^2}{\partial x^2} f_{kj}(r) = \frac{h}{d} \left(\frac{\pi k}{a}\right)^2$$
$$L_{kj}^{(2)} = -\frac{hE}{d} \int_D dr \{f_{kj}(r), \Phi[r; f_{kj}]\} f_{kj}(r) = -\frac{2^7 hE}{d} \frac{(\pi kj)^4}{(ab)^5} \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \{g_{2s+1, 2t+1} T^2_{2s+1, 2t+1, k, j}\} < 0$$

From (3.4) and (3.5) it follows that the following forms of flexure represent small appropriate solutions of (2.8):

$$\omega_*(r) \approx u_\pm(r, \; p - \Lambda_{kj}) = \pm \beta f_{kj}(r) \qquad (3.6)$$
$$\beta = [L_{kj}^{(1)}(p - \Lambda_{kj}) \, | L_{kj}^{(2)}|^{-1}]^{1/2}$$

The above forms branch off the trivial solution at the instant when the loading parameter attains the value $p = \Lambda_{kj}$. Setting in (3.2) $K = 0$, we obtain a known expression determining the moments of bifurcation of the trivial solution of the deterministic problem $(\omega_0 = 0)$.

4. The second variation of the functional $S[\omega] = -\ln(C^{-1} P[\omega])$ near $\omega = 0$ is a simple quadratic form

$$\delta^2 S[\omega] = \frac{1}{2} \sum_m c_m(p) \omega_m^2, \qquad \omega_m = \int_D dr \varphi_m(r) \omega(r) \qquad (4.1)$$

The quantities $c_m(p)$ are the eigenvalues of the operator appearing in the left-hand side of the equation

$$\int_D dr_1 \frac{\delta^2 S[\omega]}{\delta\omega(r_1) \delta\omega(r)} \bigg|_{\omega=0} u(r_1) = cu(r) \qquad (4.2)$$

The solutions of (4.2) coincide probabilistically with the zeros $\varphi_m(r)$ of the operator $B(r, p_0)$. The quantities $c_m(p)$ are greater than zero when $p < p_{0m}$, vanish when $p = p_{0m}$ and become negative when $p > p_{0m}$. This means that the form (4.1) is positive definite in the region $0 < p < p_{0n}'$ (p_{0n}' represents the smallest value of p_{0m} and is approximately equal to the smallest Λ_{st}' of Λ_{kj}), and sign definite from the moment $p = p_{0n}'$. Within the framework of approximations used, and for sufficiently small $p - (p_{0n}' \approx \Lambda_{st}') > 0$, the second variation $\delta^2 S[\omega]$ is positive in the neighborhood of the deflections (3.6): $\omega = u_\pm(r, p - \Lambda_{st}')$. It follows therefore that the probability functional $P[\omega]$ has a unique maximum at the deflection $\omega = 0$ up to the moment $p = p_{0n}' \approx \Lambda_{st}'$ when the trivial solution of (2.8) branches for the first time. The maximum passes, at this instant, into new maxima situated approximately at the deflections $\omega = u_\pm(r, p - \Lambda_{st}')$. This is an example of stochastic bifurcation. The fact that the convergence of the integral (2.4) is violated at the instant the load parameter attains the value $p = p_{0n}'$, i.e. at the instant when the stochastic system loses its stability, is a particular feature of this phenomenon. Indeed, up to the instant $p = p_{0n}'$ the functional $S[\omega]$ has a single extremum, i.e. a minimum at the deflection $\omega = 0$ ($S[\omega]$ is concave in the downward direction $\delta^2 S[\omega] > 0$) near this deflection). This implies that the functional $S[\omega]$ increases without bounds when $\omega(r) \to \pm\infty$, and the exponential term in the right-hand side of the symbolic equation

$$d\mu[\omega(r) \in \Omega] P[\omega] = d\mu[\omega(r) \in \Omega] C^{-1} \exp(-S[\omega]) \qquad (4.3)$$

is found to be, at large $|\omega(r)|$, a truncating term, and this leads to convergence of the integral (2.4) (the convergence is easily established within the framework of the perturbation theory in which, following /7/, we take the Gaussian term of the functional $P[\omega]$ as the initial approximation). In the infinitely small neighborhood of $p = p_{0n}'$ the exponential term in the right-hand side of (4.3) loses its truncating properties, and the integral (2.4) does not converge in this case. The truncating properties of (4.3) appear only when the difference $p - p_{0n}'$ exceeds the value of the infinitely small positive number ε forming the part of the divergence interval of (2.4) lying to the right of $p = p_{0n}'$. Here the integral in question again becomes convergent.

We note that the dispersion of the deflections $\omega[r; \omega_0]$ which can be determined in the region $0 < p < p_{0n}' \approx \Lambda_{st}'$ using the Gaussian approximation to the probability functional $P[\omega]$, increases sharply from below as $p \to \Lambda_{st}'$, and the characteristic dimensions of the correlations $\omega[r; \omega_0]$ approach those of the term

$$K_{st,st} f_{st}(r) f_{st}(r')$$

of the expansion of the initial deflection correlation function $K(r, r')$ over the system $\{f_{kj}(r)\}$. The deflections $\omega \approx u_{\pm}(r, p - \Lambda_{st}')$ form two basic forms of the loss of stability.

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